Strain gradient plasticity
Application to Lüders band propagation

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Plan

1. Micromorphic plasticity
   - Continuum thermodynamic formulation
   - Link to Aifantis strain gradient plasticity

2. Application to the Lüders behaviour in steels
   - Experimental evidence of Lüders band propagation
   - Mesh–dependency of standard FE simulations of Lüders bands
   - Strain gradient plasticity solution

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The micromorphic approach to elasto–plasticity (1)

(Mindlin, 1964; Eringen and Suhubi, 1964; Forest, 2009)

- The displacement and a scalar plastic microdeformation variable are the degrees of freedom $DOF = \{\underline{\mathbf{u}}\,,\; p_\chi\}$
- Define the set of state and internal variables

$$STATE = \{\underset{\sim}{\varepsilon}, \quad T, \quad p, \quad p_\chi, \quad \boldsymbol{\nabla}\; p_\chi\}$$

  the internal variable p is the accumulative plastic strain

- Extend the virtual power of internal forces

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^{\star}, \dot{p}_\chi^{\star}) = -\int_{\mathcal{D}} p^{(i)}(\underline{\mathbf{v}}^{\star}, \dot{p}_\chi^{\star})\, dV$$

$$p^{(i)}(\underline{\mathbf{v}}^{\star}, \dot{p}_\chi^{\star}) = \underset{\sim}{\sigma} : \boldsymbol{\nabla}\underline{\mathbf{v}}^{\star} + a\; \dot{p}_\chi^{\star} + \underline{\mathbf{b}}\,.\boldsymbol{\nabla}\; \dot{p}_\chi^{\star}$$

  $a, \underline{\mathbf{b}}$ generalized stresses or *microforces* (Gurtin, 1996)

- Derive additional balance equation and boundary conditions based on the method of virtual power

$$\mathrm{div}\,\underline{\mathbf{b}} - a = 0, \quad \forall \underline{\mathbf{x}} \in \Omega, \quad \underline{\mathbf{b}}\,.\underline{\mathbf{n}} = a^c, \forall \underline{\mathbf{x}} \in \partial\Omega$$

The micromorphic approach to elasto–plasticity (2)

- Enhance the local balance of energy and the entropy inequality

$$\rho\dot{\epsilon} = p^{(i)} - \mathrm{div}\,\underline{\mathbf{q}} + \rho r, \quad -\rho(\dot{\psi} + \eta\dot{T}) + p^{(i)} - \frac{\underline{\mathbf{q}}}{T}.\boldsymbol{\nabla} T \geq 0$$

- Elastic-plastic decomposition: $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^p$
- Consider the constitutive functionals:

$$\begin{aligned}
\psi &= \hat{\psi}(\underset{\sim}{\varepsilon}^e, T, p, p_\chi, \boldsymbol{\nabla} p_\chi), \ \eta = \hat{\eta}(\underset{\sim}{\varepsilon}^e, T, p, p_\chi, \boldsymbol{\nabla} p_\chi) \\
\underset{\sim}{\boldsymbol{\sigma}} &= \hat{\underset{\sim}{\boldsymbol{\sigma}}}(\underset{\sim}{\varepsilon}^e, T, p, p_\chi, \boldsymbol{\nabla} p_\chi) \\
a &= \hat{a}(\underset{\sim}{\varepsilon}^e, T, p, p_\chi, \boldsymbol{\nabla} p_\chi), \quad \underline{\mathbf{b}} = \hat{\underline{\mathbf{b}}}(\underset{\sim}{\varepsilon}^e, T, p, p_\chi, \boldsymbol{\nabla} p_\chi)
\end{aligned}$$

- State laws

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\hat{\psi}}{\partial\underset{\sim}{\varepsilon}^e}, \ \eta = -\frac{\partial\hat{\psi}}{\partial T}, \ R = \rho\frac{\partial\hat{\psi}}{\partial p}, \quad a = \frac{\partial\hat{\psi}}{\partial p_\chi}, \quad \underline{\mathbf{b}} = \frac{\partial\hat{\psi}}{\partial\boldsymbol{\nabla} p_\chi}$$

- Residual dissipation $D^{res} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}}^p - R\dot{p} - \frac{\underline{\mathbf{q}}}{T}.\boldsymbol{\nabla} T \geq 0$

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Micromorphic elasto–plasticity

- Quadratic free energy potential

$$\rho\psi(\underset{\sim}{\varepsilon}^e, p,\ p_\chi, \nabla p_\chi) = \frac{1}{2}\underset{\sim}{\varepsilon}^e : \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e + \frac{1}{2}Hp^2 + \frac{1}{2}H_\chi(p-p_\chi)^2 + \frac{1}{2}\nabla p_\chi.\underset{\sim}{\mathbf{A}}.\nabla p_\chi$$

- Constitutive equations

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e,\ a = -H_\chi(p-p_\chi),\ \underline{\mathbf{b}} = \underset{\sim}{\mathbf{A}}.\nabla p_\chi,\ R = (H+H_\chi)p - H_\chi p_\chi$$

- Substitution of constitutive equation into the extra–balance equation

$$p_\chi - \frac{1}{H_\chi}\mathrm{div}\left(\underset{\sim}{\mathbf{A}}.\nabla p_\chi\right) = p$$

- Homogeneous and isotropic material $\underset{\sim}{\mathbf{A}} = A\underset{\sim}{\mathbf{1}}$

$$p_\chi - \frac{A}{H_\chi}\Delta p_\chi = p, \quad \text{b.c.} \quad \nabla p_\chi.\underline{\mathbf{n}} = a^c$$

Link to Aifantis strain gradient plasticity

- Yield function $$f(\underset{\sim}{\sigma}, R) = \sigma_{eq} - \sigma_Y - R$$
- Evolution laws $$D^{res} = \underset{\sim}{\sigma} : \underset{\sim}{\dot{\varepsilon}}^p - R\dot{p} - X\dot{\alpha} \geq 0$$

$$\underset{\sim}{\dot{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \underset{\sim}{\sigma}}, \quad \dot{p} = -\dot{\lambda}\frac{\partial f}{\partial R}$$

- Hardening law

$$R = \frac{\partial \psi}{\partial p} = (H + H_\chi)p - H_\chi p_\chi$$

- Under plastic loading

$$\sigma_{eq} = \sigma_Y + Hp_\chi - A(1 + \frac{H}{H_\chi})\Delta p_\chi$$

compare with Aifantis model (Aifantis, 1987)

$$\sigma_{eq} = \sigma_Y + R(p) - c^2\Delta p$$

The equivalence is obtained for $H_\chi = \infty$ (internal constraint):

$$p_\chi \simeq p, \quad A = c^2$$

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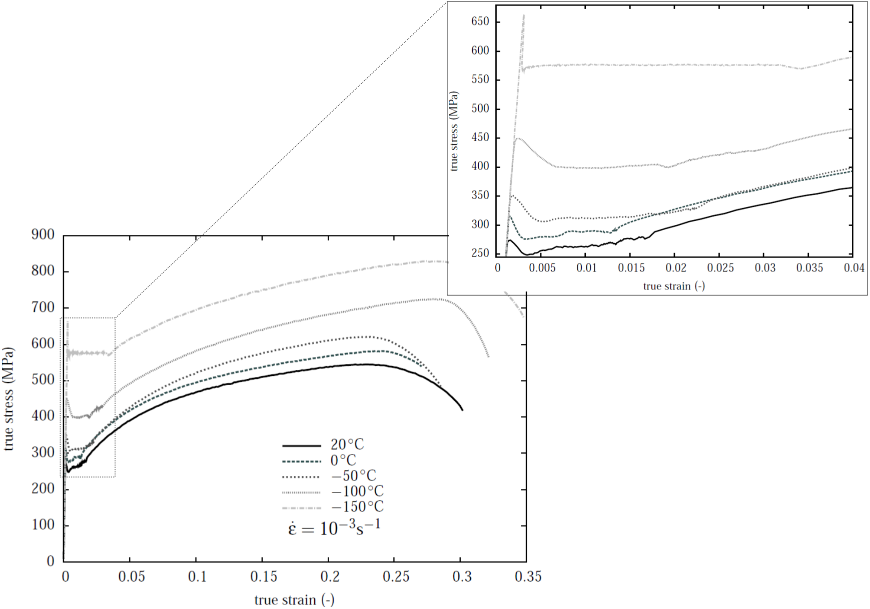
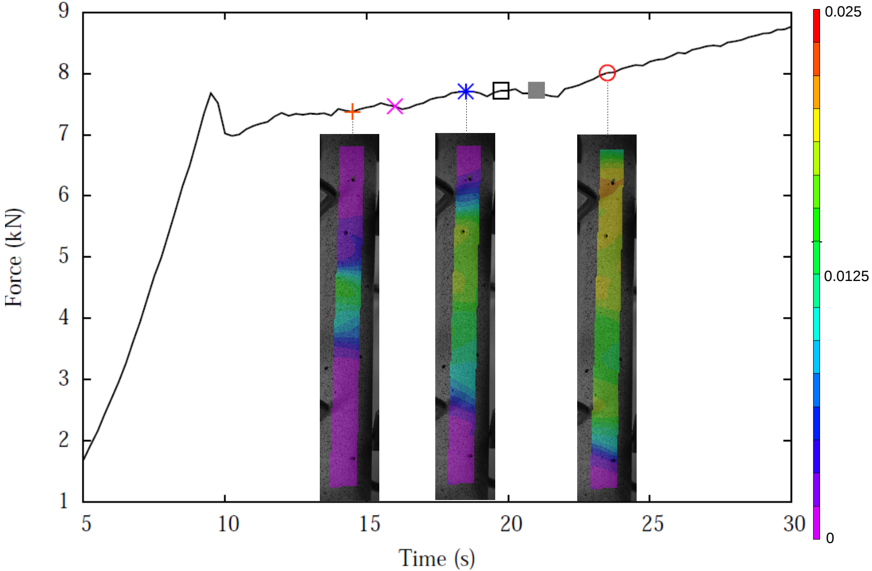
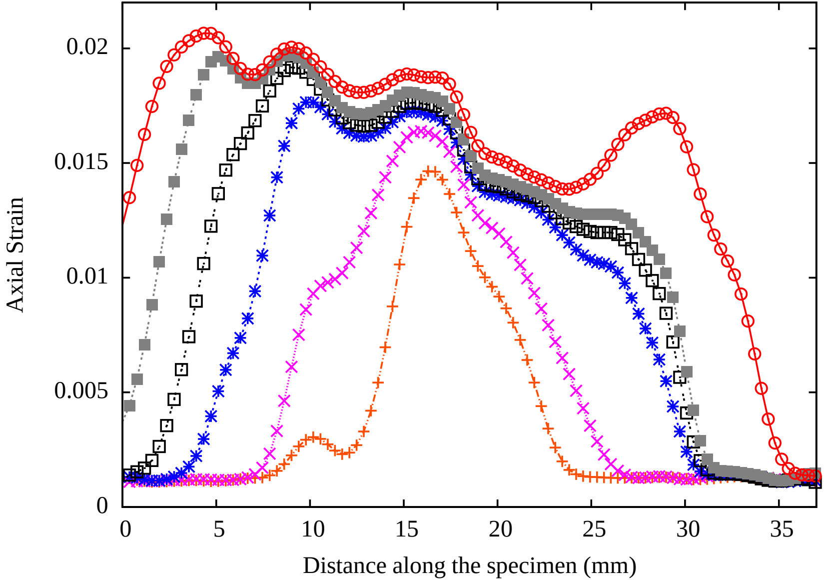

true stress (MPa)
true strain (-)
20°C
0°C
−50°C
−100°C
−150°C
$\dot{\varepsilon} = 10^{-3} s^{-1}$

Strain field measurements



Strain field measurements



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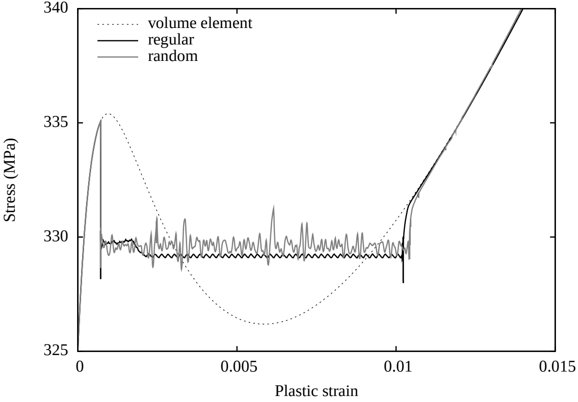
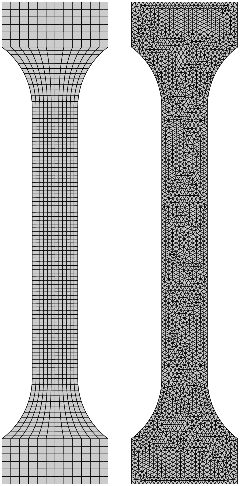
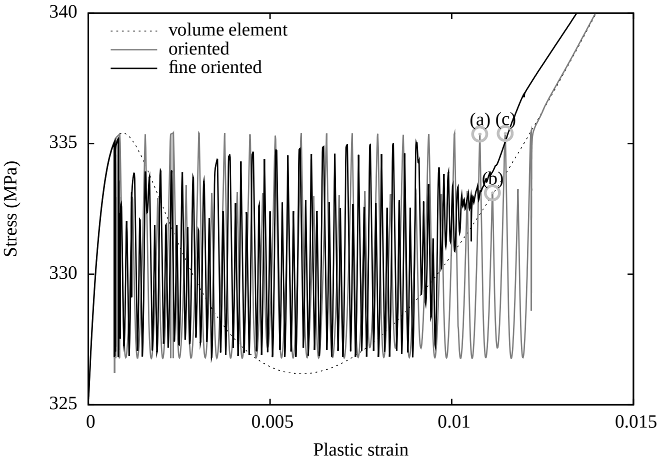
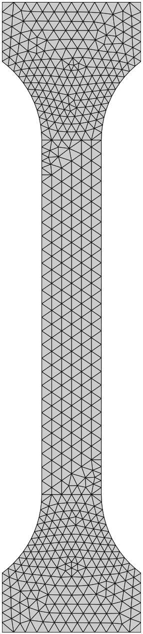


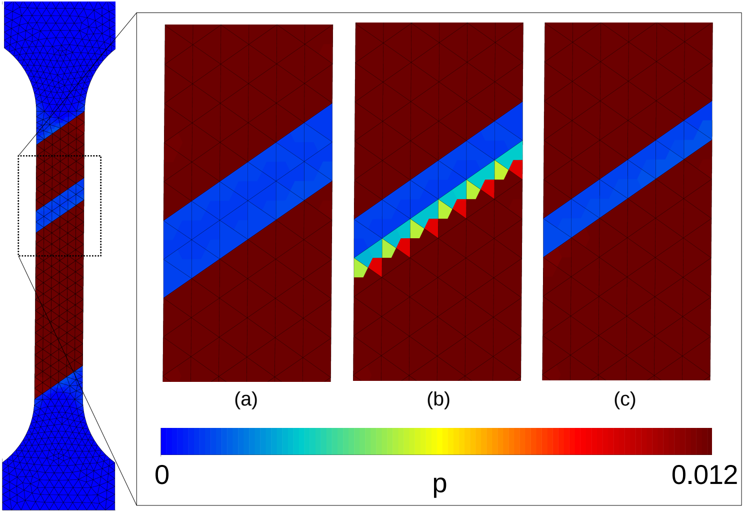
340
335
330
325
Stress (MPa)
volume element
regular
random
0
0.005
0.01
0.015
Plastic strain





340
335
330
325
0
0.005
0.01
0.015
volume element
oriented
fine oriented
(a)
(b)
(c)
Stress (MPa)
Plastic strain





(Ballarin et al., 2009)

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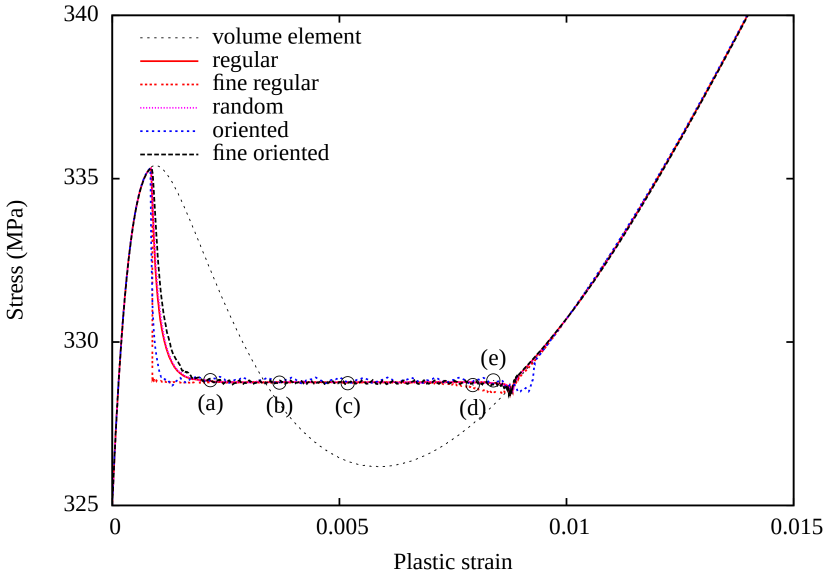
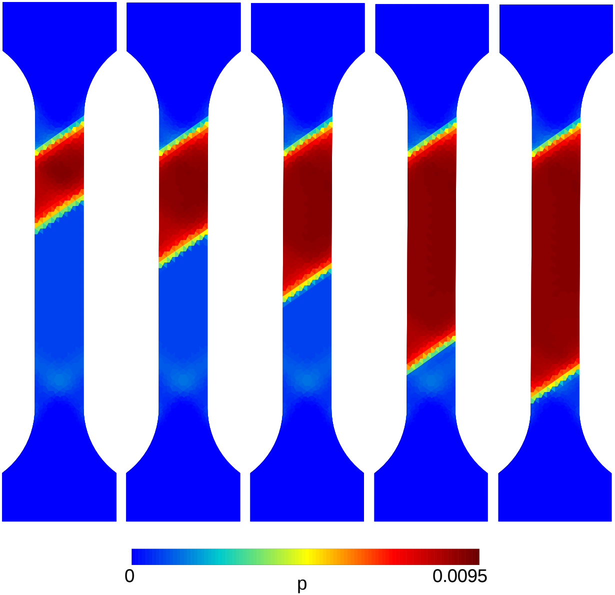
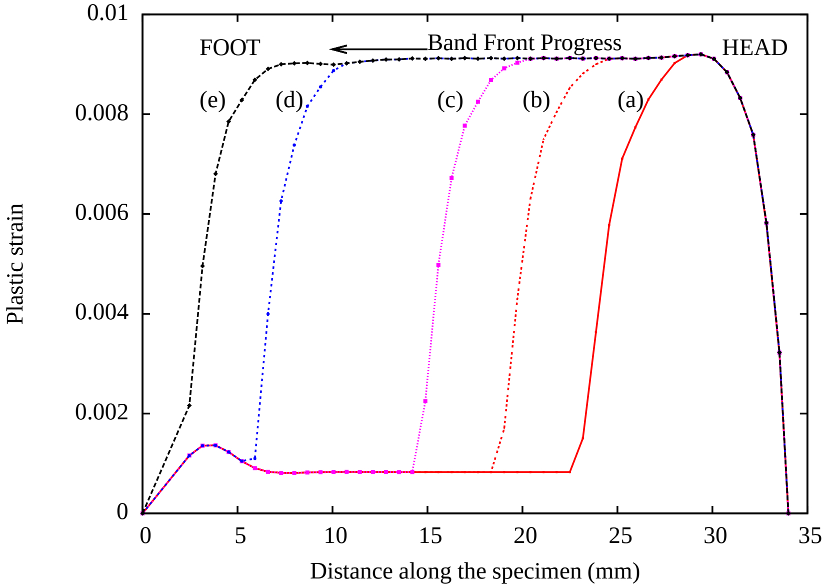


volume element
regular
fine regular
random
oriented
fine oriented
(a)
(b)
(c)
(d)
(e)
Stress (MPa)
340
335
330
325
0
0.005
0.01
0.015
Plastic strain

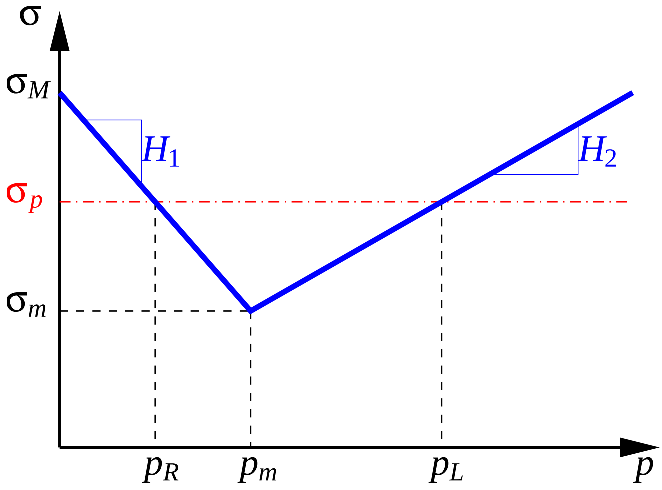
Smooth propagation of the front



Smooth propagation of the front

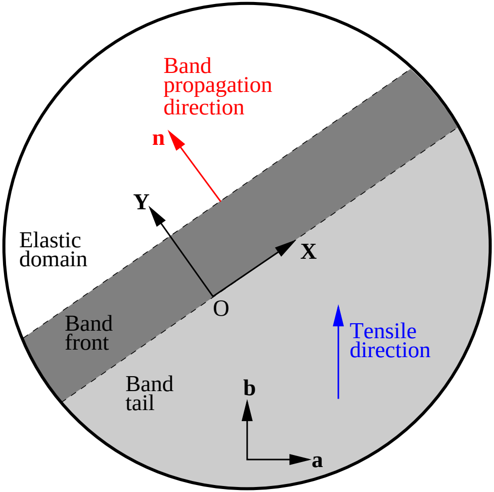


Multi–linear softening-hardening material



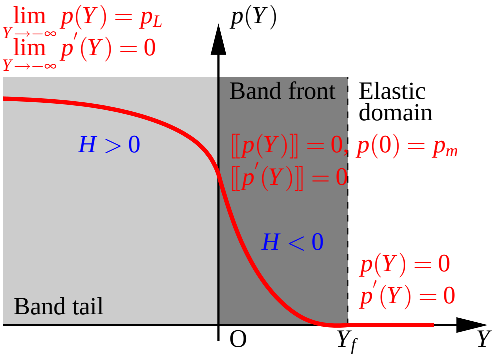
peak stress: σ_M, minimal stress: σ_m, plateau stress σ_p, Lüders strain p_L, hardening moduli $H_1 < 0, H_2 > 0$.

Bifurcation analysis



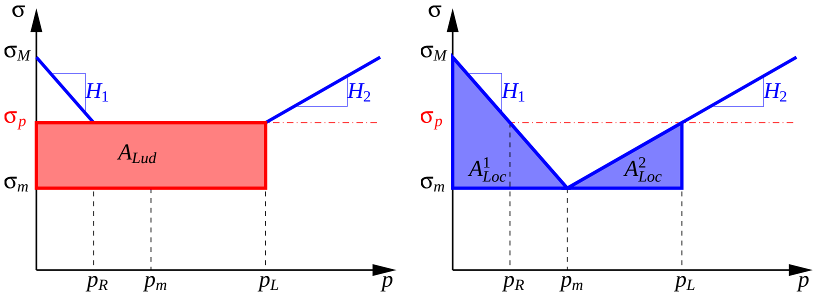
Homogeneous tensile stress state. The strain localization band in 2D (Rice's criterion) is inclined at 54.7° from the the tensile axis

Description of the band front

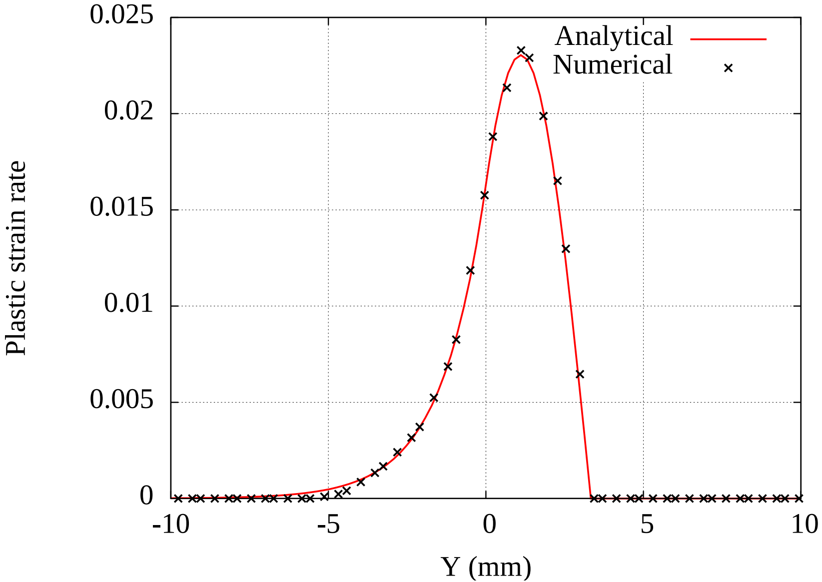


- $\sigma_p = \sigma_m + H_2(p - p_m) - Ap''$, $\quad l_2^2 = \dfrac{A}{H_2} \Longrightarrow$ sine branch
- $\sigma_p = \sigma_m + H_1(p - p_m) - Ap''$, $\quad l_1^2 = -\dfrac{A}{H_1} \Longrightarrow$ hyperbolic sine branch
- interface conditions

Maxwell's rule



determination of the plateau stress and Lüders strain



Analytical and finite element plastic strain rate profiles $\dot{p}$

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